GAMES AGAINST NATURE

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CONTENTS

Summary: An axiomatic study is made of various criteria for playing games against Nature; and a new criterion is suggested. The concept of equilibrium point is defined for n-person games against Nature.

GAMES AGAINST NATURE

John Milnor

Introduction.

We wish to study games of the following type. A matrix is given in which a player must choose a row. A column will be chosen by "Nature", whose payoff is unknown or identically zero. The payoff to the player is given by the entry in that particular row and column. This entry should represent a von Neumann-Morgenstern utility. It is of course also interesting to study the case of an infinite game, but this will not be attempted here.

The following criteria have been suggested for this situation. According to Laplace, if the relevant probabilities are unknown, we should assume that they are equal. In other words, the player should choose the row for which the average of the elements is greatest.

According to the minimax principle which is associated with Wald, we should assume that the payoff to Nature is the negative of the payoff to the player. The solution is then just the solution of the matrix considered as the matrix of a zero sum game.

A modification of this criterion was suggested by Hurwicz. Select a constant $0 \leq \alpha \leq 1$ which measures the player's optimism. For each row or probability mixture of rows let m denote the

smallest component and M the largest.. According to Hurwicz we
should select one of those rows, or mixtures of rows, for which
$\alpha M + (1-\alpha)m$ attains its maximum. For $\alpha = 0$ this reduces to
the Wald criterion.

A different modification of the minimax principle is given by
Savage, who holds that the payoff to Nature should be the "regret",
"loss", or "miss" which is measured by the maximum of the elements
of a column minus the particular entry. In other words, the regret
is the difference between what could have been gotten if the state
of Nature were known and what actually was gotten. A strategy is
to be chosen as if the regret matrix were that of a zero sum game.

The present paper will attempt to study these criteria and
suggest others on the basis of an axiomatic approach.

§I. Axioms for the Laplace, Wald, and Hurwicz criteria.

It will be convenient to make no distinction between rows and
probability combinations of rows. We will first consider criteria
which assign a preference relation $\gtrsim$ between pairs of rows, or
between pairs of probability combinations of rows, satisfying the
following axioms.

1.1 The relation $\gtrsim$ is a complete ordering. In other words,
the following two laws are satisfied.

1.1a For any r and r' either $r \gtrsim r'$ or $r' \gtrsim r$.

1.1b If $r \gtrsim r' \gtrsim r''$ then $r \gtrsim r''$.

1.2 The order of two rows is not changed by the adjunction
of a new row.

1.3   It is not changed by any permutation of the rows or columns.

1.4   If each element of r is greater than the corresponding element of r' then $r > r'$.

The first of these axioms is somewhat stronger than is necessary, since we only need to know the set of optimal strategies and not the complete ordering relation. Axiom 1.2 seems much too strong to me. Part of my object in writing section I is to show that this axiom is incompatible with others which seem more fundamental to me. The criterion of Savage violates this axiom and therefore is not considered in this section. Axiom 1.3 is actually a part of the description of the problem. That is, we are not considering games in which there is any reason to believe that one column is more probable than another. Axiom 1.4 has been put in a weaker form than is usual since the Wald, Hurwicz, and Savage criteria do not satisfy the stronger condition that $r > r'$ whenever r dominates r'.

Some of the following axioms may also be imposed.

1.5   The order of the rows is not changed if a constant is added to a column.

1.6   It is not changed if a new column c' is added, if $c' = a_1 c_1 + \cdots + a_n c_n$ where $a_i \geq 0$, $a_1 + \cdots + a_n = 1$.

1.7   It is not changed if each element P_{ij} of the matrix is replaced by $\lambda P_{ij} + \mu$, $\lambda > 0$.

1.8   The ordering relation is continuous in the following sense. If a sequence of matrices P_i converges to P and if for all i,

$r_i \underset{\sim}{\geq} r_i'$, then the limit rows r and r' satisfy $r \underset{\sim}{\geq} r'$.

 1.9 The set of optimal strategies is convex.

 Axiom 1.5 is in part a statement that there is no reason to believe that nature is against us. It is also a statement that the utility is linear, not only with respect to known probabilities, but also with respect to unknown probabilities of the type under consideration. It is satisfied by the Laplace and Savage criteria. Axiom 1.6 is satisfied by all but the Laplace criterion, while 1.7 and 1.8 are satisfied by all four. Axiom 1.9 states that it is irrational to be equally willing to play two alternatives and yet not be willing to randomize between them. It is violated by the Hurwicz criterion for $\alpha > 0$ (in the matrix $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ for example).

 The following assertions will be proved in the appendix. Axioms 1.1 through 1.4 are implicitly assumed.

 1.10 If two rows differ only in the order of their elements, then they are equivalent.

 1.11 The only criterion satisfying 1.5 is that of Laplace.

 1.12 For any criterion satisfying 1.6, two rows having the same maximum and minimum elements are equivalent.

 1.13 Axioms 1.6, 1.7, and 1.8 imply the Hurwicz criterion.

 1.14 Axioms 1.6, 1.8, and 1.9 imply the Wald criterion.

§II. Alternative Axioms.

 Since the nine axioms considered in section I have been shown to be inconsistent, it is natural to try and pare them down to a

set of more fundamental axioms. The following is a list of those axioms which I personally feel to be most important. Others would doubtlessly set up completely different lists. Let S denote the simplex of mixed strategies over the rows.

2.1   There is a choice set $C(P_{ij})$ which is contained in S.

2.2   This set depends continuously on P_{ij} in the following sense. If P_{ijk} approaches P_{ij}, $x_k \in C(P_{ijk})$ for each k, and x_k approaches x, then $x \in C(P_{ij})$. In particular C is a closed set.

2.3   C is convex.

2.4   The choice set is invariant under permutations of the rows or columns.

2.5   Every element of C is admissible (undominated).

2.6   The choice set is not changed if a new column, identical with one of the old, is added.

These axioms are not compatible with any of the criteria under consideration. That of Laplace violates 2.6, while those of Wald, Hurwicz, and Savage violate 2.5. It is of course possible to intersect the choice set of Wald, Hurwicz, or Savage with the set of admissible strategies, and thus obtain a new criterion. The result violates 2.2 however. The Hurwicz criterion also violates axiom 2.3, providing that $\propto > 0$.
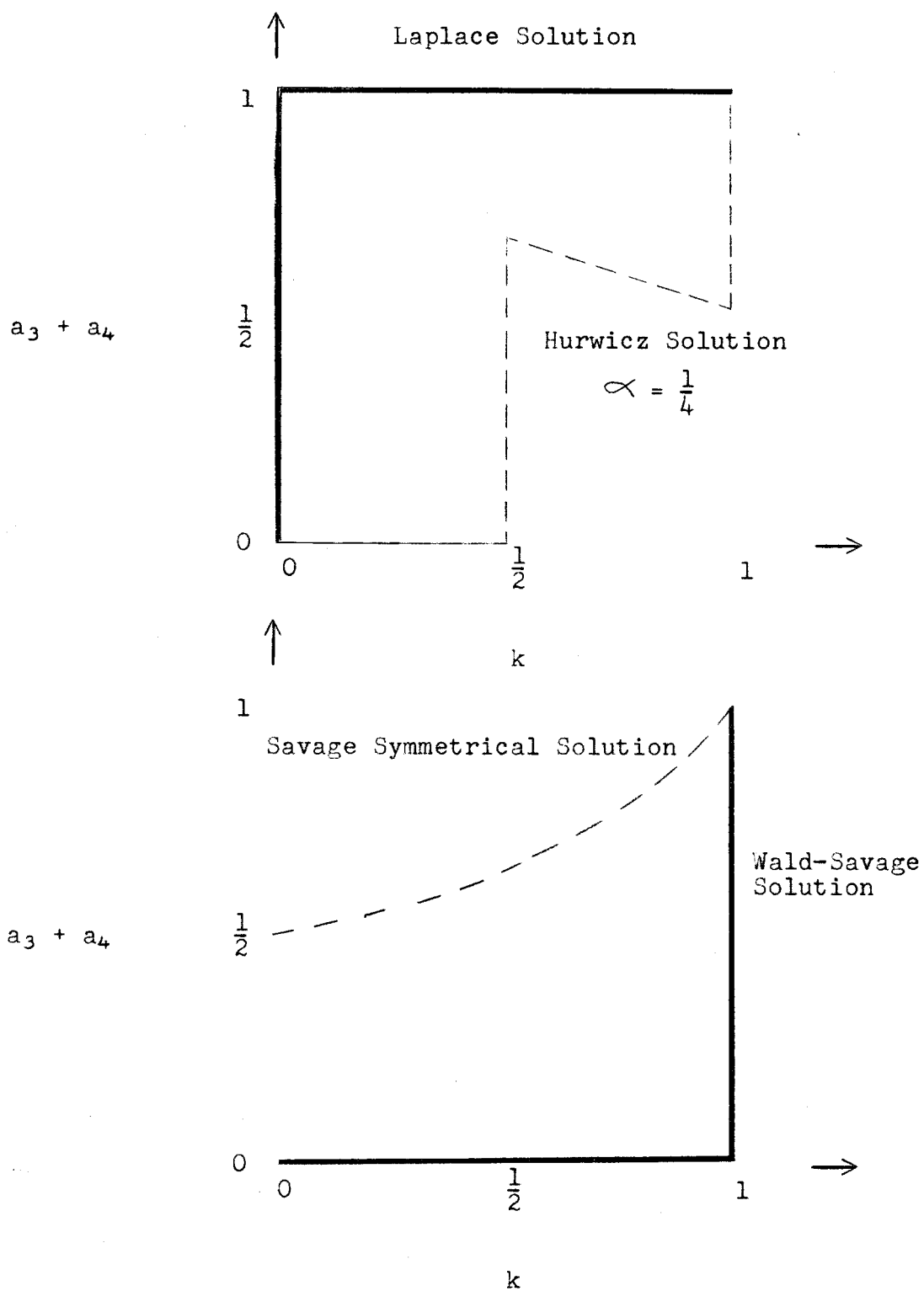
As an example of the discrepancy between these axioms and the four criteria consider the following family of games, where $0 \leq k \leq 1$.

$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & k & 1 \\ 1 & k & 0 & 1 \end{pmatrix}$$

Let a_i denote the probability of playing the ith row for
$i = 1, 2, 3, 4$. For k=1, each of the first two rows is dominated
by one of the last two. For any criterion satisfying 2.5, there-
fore, each point of the choice set must satisfy $a_3 + a_4 = 1$. By
continuity $a_3 + a_4$ must be close to one for k close to one.
Solutions by the four criteria, and graphs of these solutions
follow. It may be that a player who uses the Savage criterion will
take the symmetries of the matrix into account and consider only
symmetrical strategies (that is strategies with $a_1 = a_2$, $a_3 = a_4$).
The solution in this case is also included below, since it illus-
trates the dependence of the Savage solution on the set of strat-
egies considered.

Criterion	Game	a_1	a_2	a_3	a_4
Admissibility	$k<1$	. . . . anything			
	$k=1$	0	0	$a_3 + a_4 = 1$	
Laplace	$k=0$	. . . . anything			
	$0<k$	0	0	$a_3 + a_4 = 1$	
Wald and Savage	$k<1$	$\frac{1}{2}$	$\frac{1}{2}$	0	0
	$k=1$	. . . $a_1 = a_2$ . . . . . . $a_3 = a_4$ . . .			
Savage Symmetrical Solutions Only		$\frac{1-k}{2(2-k)}$	$\frac{1-k}{2(2-k)}$	$\frac{1}{2(2-k)}$	$\frac{1}{2(2-k)}$
Hurwicz $\alpha<1$	$k<1-2\alpha$	$\frac{1}{2}$	$\frac{1}{2}$	0	0
	$1>k>1-2\alpha$	0	$\frac{k}{1+k}$	$\frac{1}{1+k}$	0
		. . . . or			
		$\frac{k}{1+k}$	0	0	$\frac{1}{1+k}$
	$k=1-2\alpha$	$0\leq\alpha_1\leq\frac{1}{2}$	$\frac{a_1+k-ka_1}{1+k}$	$\frac{1-2a_1}{1+k}$	0
		. . . . or			
		$\frac{a_2+k-ka_2}{1+k}$	$0\leq a_2\leq\frac{1}{2}$	0	$\frac{1-2a_2}{1+k}$
	$k=1$	0	$a_2+a_4=\frac{1}{2}$	$\frac{1}{2}$	
		. . . . or			
		$a_1+a_3=\frac{1}{2}$	0		$\frac{1}{2}$
$\alpha=1$		$\alpha_1 = 0$ or $\alpha_2 = 0$			

GRAPHS OF SOLUTION

Laplace Solution

$a_3 + a_4$

1

$\frac{1}{2}$

Hurwicz Solution

$\propto = \frac{1}{4}$

0

0    $\frac{1}{2}$ 1

k

1

Savage Symmetrical Solution

Wald-Savage
Solution

$a_3 + a_4$

$\frac{1}{2}$

0

0    $\frac{1}{2}$ 1

k

§III. Other Criteria.

A general class of decision criteria related to that of Laplace are those which construct a linear "utility" function on the simplex S. The choice set is then the subsimplex of points which maximize this function.

Since we are only interested in differences of utility, there is no need to distinguish between functions which differ by a constant. This identification of functions is best accomplished by considering changes in strategy instead of fixed strategies. For any linear function u and change of strategy $\triangle s$ the function

$$u^*(\triangle s) = u(s+\triangle s) - u(s)$$

is independent of s, and does not change if a constant is added to u. We wish, therefore, to study linear function which are defined for all $\triangle s$ and which vanish for $\triangle s = 0$. Let $\triangle$ denote the space of strategy changes $\triangle s$. If an element of S is represented by coordinates $(a_1, \ldots, a_m)$, $a_1 + \cdots + a_m = 1$, $a_i \geq 0$; then an element of $\triangle$ is represented by $(\alpha_1, \ldots, \alpha_m)$ $\alpha_1 + \cdots + \alpha_m = 0$, such that $\alpha_{i_1} + \cdots + \alpha_{i_k} \leq 1$ for all sets of distinct $i_1, \ldots, i_k$. The set of linear functions on $\triangle$ which vanish at the origin form a vector space which may be denoted by U.

Corresponding to each mixed strategy for Nature, that is each probability combination of columns, there is a linear function

on S and therefore a point in U. If Nature plays the column

$c = (c_1, \ldots, c_m)$ and the change of strategy $\delta = (\alpha_1, \ldots, \alpha_m)$

is made by the player, the gain to the player is the inner product

$$c \cdot \delta = c_1 \alpha_1 + \cdots + c_m \alpha_m.$$

This product defines the linear function on $\triangle$ corresponding to

this strategy of Nature. The set of all points which correspond

to mixed strategies for Nature forms a convex set $N \subseteq U$. It seems

reasonable that the function we should construct should be a

point of N so that the solution will be a Bayes solution. This

condition is equivalent to the following.

3.2 For any pair of rows or probability mixtures of rows r

and r', the linear function u satisfies

$$u(r-r') \leq \underset{j}{\text{Max}} \ (r_j - r_j')$$

where j runs over all columns.

In particular if each component of r is less than the cor-

responding component of r' then $u(r) < u(r')$, which is just

axiom 1.4.

In this context the problem of choosing a decision criteria

becomes simply the problem of defining a unique point in each con-

vex set N in some suitable way. Since we wish to assume some

type of average behavior for Nature, this point may be considered

as a center for N. The Laplace criterion is that obtained by

taking the center of gravity of the vertices of N, the weight of

each vertex being the number of times the corresponding column

appeared in the matrix. An obvious modification of this would be

to take the center of gravity of the convex set N; but this is

unfortunately discontinuous when N changes dimension.

One especially simple case occurs when N is a line segment.

Any reasonable definition for center will then specify the midpoint

of N. This occurs in two cases. If the original matrix has two

columns, then these columns generate a line segment, the center of

which gives the Laplace solution. If, on the other hand, the

matrix has two rows, then the space of linear functions, and there-

fore N will again be one dimensional. The midpoint in this case

corresponds to the following rule.

3.1 In the matrix $\begin{pmatrix} a_1 & \cdots & a_n \\ b_1 & \cdots & b_n \end{pmatrix}$ choose the first or second

row according as $\underset{j}{\text{Max}} \ (a_j - b_j)$ is greater than or less than

$\underset{j}{\text{Max}} \ (b_j - a_j)$. In case equality holds, choose either one or any

mixture.

This may also be formulated as follows. Assume that Nature

will play that column for which the player's decision makes the

most difference.

One possible procedure for defining such a center for N will

be given. Many others are possible, however, and I have no strong

justification for this one. It is motivated by the following idea.

If a decision procedure of this type cannot satisfy axiom 1.2, we

can at least require that it come as close as is possible to satis-

fying it. In particular we can require that for every two row sub-

matrix, the utility of changing from one row to another should be

as close as possible to the utility of making this change in the
full matrix. For utility function $u \in N$ the utility of making
the change δ is $u(\delta)$. In the two row submatrix, the utility
of changing from $r = (a_1, \ldots, a_n)$ to $r' = (b_1, \ldots, b_n)$, using
the midpoint solution just mentioned, is

$$f(\delta) = \frac{1}{2} \max_j (a_j - b_j) + \frac{1}{2} \min_j (a_j - b_j).$$

We wish therefore to find which $u \in N$ can be considered as the
best approximation to f.

Now since f is defined in terms of the operations Max and
Min applied to a finite number of linear functions over $\triangle$,
there is some polyhedral decomposition of $\triangle$ such that the function
f, and therefore $u-f$, is linear on each polyhedron of the decompo-
sition. It is therefore sufficient to know the values of $u-f$ on
the vertices of these polyhedrons in order to tell how well u
approximates f. Specify the function u by the following three
requirements.

3.2 The maximum value of $|u-f|$ shall be as small as possible.

3.3 The set of vertices on which this minimax value is attained
shall be as small as possible.

This last condition is meaningful since, if u_1 and u_2 attain
the minimax value on different sets, then $\frac{u_1 + u_2}{2}$ attains this value
at most on the intersection of these two sets. Therefore by aver-
aging a finite number of such functions we obtain a function which
attains this value on a smallest possible set of vertices. The set

of $u \in N$ which satisfy these two conditions forms a convex set, whose closure may be denoted by N_1. Let v_1 denote the set of vertices at which these functions take on values less than this maximum. We have thus proceeded from the set $N=N_0$ and the set v_0 of all vertices to smaller sets having the same properties. We now proceed inductively.

3.4 Given sets N_i and v_i take the set N_{i+1} of limit points of functions satisfying 3.2 and 3.3 and the set v_{i+1} of vertices where the maximum value need not be attained. Repeat until v_{r+1} is vacuous. The set N_{r+1} will then consist of a single point which is the required function u.

The criterion which is specified by this rule satisfies all of the axioms of section II, and all except 1.2 of section I. The following simple example will be worked out to illustrate the rule.

$$\begin{pmatrix} 5 & 1 & 3 \\ 1 & 4 & 2 \\ 4 & 1 & 4 \end{pmatrix}$$

If an element of $\triangle$ is represented by a triple (α, β, γ), then the vertices of $\triangle$ (which is a hexagon) are $(1,-1,1)$. The function f is derived from the three functions

$$5\alpha + \beta + 4\gamma \qquad \alpha + 4\beta + \gamma \qquad 3\alpha + 2\beta + 4\gamma$$

by the operations Max and Min. It is therefore nonlinear whenever

two of these functions are equal. This defines the three lines

$$5\alpha + \beta + 4\gamma = \alpha + 4\beta + \gamma \text{ or } 4\alpha + 3\gamma = 3\beta,$$
$$2\alpha = \beta, \text{ and } 2\alpha + 3\gamma = 2\beta.$$

The vertices of the polyhedrons formed by these lines and $\triangle$, other than $(0,0,0)$ and the six already mentioned are $(-\frac{6}{7},-\frac{1}{7},1)$, $(\frac{6}{7},\frac{1}{7},-1)$, $(\frac{1}{3},\frac{2}{3},-1)$, $(-\frac{1}{3},-\frac{2}{3},1)$, $(1,-\frac{1}{5},-\frac{4}{5})$, $(-1,\frac{1}{5},\frac{4}{5})$. Listing one vertex from each pair consisting of a vertex and its negative, we have the following table.

Vertex σ	Max $c \cdot \sigma$ c	Min $c \cdot \sigma$ c	f
$(1,-1,0)$	4	-3	$\frac{1}{2}$
$(1,0,-1)$	1	-1	0
$(0,1,-1)$	3	-3	0
$(-\frac{6}{7},-\frac{1}{7},1)$	$1\frac{1}{7}$	$-\frac{3}{7}$	$\frac{5}{14}$
$(\frac{1}{3},\frac{2}{3},-1)$	2	$-\frac{5}{3}$	$\frac{1}{6}$
$(1,-\frac{1}{5},-\frac{4}{5})$	$1\frac{3}{5}$	$-\frac{3}{5}$	$\frac{1}{2}$

If $u(\alpha,\beta,\gamma) = a\alpha + b\beta$, then the vertices of N are the three points $(a,b) = (1,-3)$, $(0,3)$, and $(-1,-2)$. The point $(a,b) = (\frac{1}{15},-\frac{1}{3})$ is in the triangle N and is a best approximation to f, using 3.2 alone. The values of u are shown in the following table.

$f(\sigma)$	$u(\sigma)$	$|f-u|$
$\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{10}$
0	$\frac{1}{15}$	$\frac{1}{15}$
0	$-\frac{1}{3}$	$\frac{1}{3}$
$\frac{5}{14}$	$-\frac{1}{105}$	$\frac{11}{30}$
$\frac{1}{6}$	$-\frac{1}{5}$	$\frac{11}{30}$
$\frac{1}{2}$	$\frac{2}{15}$	$\frac{11}{30}$

Using this function u as utility function, we find that it is worth $\frac{1}{15}$ to switch from row 3 to 1; and worth $\frac{6}{15}$ to switch from 2 to 1. The choice set is therefore row 1.

I have not computed the example of section II in detail, but it seems that the solution is identical with the Laplace solution in this case. The function u depends on k in a fairly complicated way, however.

§IV. n-person Games.

Consider a game of the following type. Each of n persons makes a move s_i out of a set of possible moves S_i, $i = 1, \ldots, n$. Simultaneously, nature makes a move s_0 out of a set of possible moves S_0. The payoff $P_i(s_0, s_1, \ldots, s_n)$ is defined for $i = 1, \ldots, n$. As an example of this consider a game in which the players have incomplete information as to the rules. This may be treated as a case in which nature chooses a particular set of

rules s_0 out of a set of possibilities.

Suppose that each player has some criterion which he believes in for games against nature. Then if each player except i has made a choice of strategy beforehand, the ith player is playing a one person game against nature, and can apply his criterion. By an <u>equilibrium point</u> of such a game will be meant a set of strategies $s_1, \ldots, s_n$ possessing the following property. For each i, if players 1, ..., i-1, i+1, ..., n hold their strategies fixed, then according to player i's criterion s_i is one of the optimal strategies in the resulting game between player i and nature. This is a generalization of the definition of equilibrium point for ordinary n-person games which was given by John Nash.

<u>Theorem</u>. <u>If each player has a criterion satisfying axioms 2.1, 2.2, and 2.3 then there exists at least one equilibrium point.</u>

The proof by the Kakutani fixed point theorem follows Nash's proof.

In most applications different players will have different amounts of information as to what nature will do. The game which we have considered does not take this into account as it stands; but this game can be considered as a normal form for a much wider class of games. This reduction to normal form is illustrated by some of the examples which follow.

We will consider variations in the following game which was suggested by E. W. Paxson. It is modeled after a "no limit" poker game. Players I and II have resources $M > 0$ and $N > 0$ respectively. Each knows his own resources but may have incomplete

information as to his opponent's. Each has a choice of matching

or not matching. If they match, the one with higher resources

wins the resources of the other. In case of a tie, no exchange

is made. If one player fails to match, he is penalized half his

resources, and the other player collects this amount. If both

fail to match, each is penalized half his resources and no one

collects. If 0 denotes matching and 1 denotes not matching,

we have the following matrix.

	$y = 0$	$y = 1$
$x = 0$	$N,-N$ if $M > N$ $0,0$ if $M = N$ $-M,M$ if $M < N$	$\dfrac{N}{2},-\dfrac{N}{2}$
$x = 1$	$-\dfrac{M}{2},\dfrac{M}{2}$	$-\dfrac{M}{2},-\dfrac{N}{2}$

The equilibrium points, under the assumption that both players use

the Savage criterion, are given below.

    4.1 Each player knows his own resources but has no information

about those of his opponent. In this case there are two equilibrium

points. One at $x=0$, $y=\frac{1}{4}$ and the symmetric one at $x=\frac{1}{4}$, $y=0$.

This non-unique answer seems to be caused by the unboundedness of

the set of possible resources, rather than by the non-zero sum nature

of the matrix. This is illustrated by the following case.

    4.2 Each player is told only his own resources, but both are

given the information that $0 < M < 1$, $0 < N < 1$. There is now a

unique equilibrium point at

$$x = \text{Min}\left\{ .1569, \ .5524 + \frac{.0262}{M} - \frac{\sqrt{.0246 + 1.040M - 1.019M^2}}{6M} \right\}$$

where y is the same function of N. The graph of this function
looks roughly as follows.
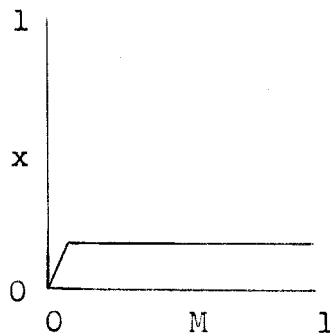
The motivation for the greater tendency to match when the resources
become very small, is simply that there is very little to lose in
this case and a great deal to gain in case the opponent passes with
larger resources.

4.3 The resources are again allowed to be arbitrarily large.
Each player is told his own resources and given an upper and lower
bound for the resources of his opponent. In other words Nature
chooses the six numbers

$$0 < N' < N < N'' \quad \text{and} \quad 0 < M' < M < M''.$$

Player I is told M,N', and N". Player II is told N,M', and M".
The matrix is as before. In this case there is again a unique
equilibrium point in which the two players have the same strategy.

This strategy is given by

$$x = \begin{cases} 0 \text{ for } N'' \le M \\[4pt] \frac{1}{4} \text{ for } N' < M < N'' \le 14M \\[4pt] \frac{4M}{N''+2M} \text{ for } M \le N' \text{ and } 2M \le N'' \\[4pt] \quad \text{or} \quad \text{for } 14M \le N'' \\[4pt] 1 \text{ for } M \le N' \text{ and } N'' \le 2M \end{cases}$$

A graph of x as a function of $\frac{M}{N''}$ follows. The upper branch is for $N' \ge M$ and the lower branch for $N' < M$.
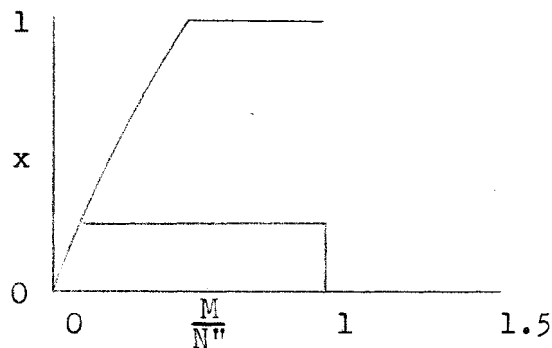
The solutions to this example by the criteria of section III are much simpler. I do not know how to extend this criterion to infinite games in general. For these particular examples, however, since there are only two alternatives for the player, the solution may be obtained from 3.1.

For the games 4.1 and 4.2 the solution is to match (play x=0) in all cases. For 4.3 the solution is to match whenever $N' < M$, that is, whenever it is possible that $N \le M$, and to pass otherwise.

APPENDIX

I. Proofs of assertions in section 1.

1.10 Suppose $(P_1 P_2 P_3 \cdots P_n) > (P_2 P_1 P_3 \cdots P_n)$. Interchanging

the first two columns we have $(P_2 P_1 P_3 \cdots P_n) > (P_1 P_2 P_3 \cdots P_n)$

which is a contradiction. Since any permutation of the elements can

be achieved by successive interchanges of pairs of elements, the

conclusion follows.

1.11 Suppose the average of the elements of r equals the

average of the elements of r'. Alternately perform the following

operations on the two row submatrix formed by r and r'.

      a) Permute the elements of each row so that they are in

order of increasing size.

      b) Subtract from each column the minimum of its element.

After a finite number of steps, all of the components of the matrix

will be zero, and therefore $r \sim r'$.

It follows by 1.4 that if the average of the elements of r is

greater than the average of the elements of r', then r > r'.

1.12 It is sufficient to show that if $m \leq P_i \leq M$ then

$(m, m, \ldots, m, M)$ is equivalent to $(m, P_1, \ldots, P_{n-2}, M)$. Alternately

applying 1.10 and 1.6 we have the following sequence of pairs of

equivalent rows.

$$\begin{pmatrix} m & M \\ m & M \end{pmatrix} \longrightarrow \begin{pmatrix} M & m \\ m & M \end{pmatrix} \longrightarrow \begin{pmatrix} M & M & m \\ m & m & M \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} m & M & M \\ m & m & M \end{pmatrix} \longrightarrow \begin{pmatrix} m & P_1 & \cdots & P_{n-2} & M & M \\ m & m & \cdots & m & m & M \end{pmatrix}$$

The next to the last column may be removed by reversing all but the last step.

1.13 By 1.12 it is sufficient to consider pairs (M_i, m_i) satisfying $M_i \geq m_i$ in place of rows. Let α_i be the supremum of α such that

$$(\alpha M_i + (1-\alpha)m_i, \; \alpha M_i + (1-\alpha)m_i) \lesssim (M_i, m_i) \text{ and } 0 \leq \alpha \leq 1.$$

By axiom 1.7 the α_i which is obtained is independent of i providing only that $M_i > m_i$. Denote it by α. By axiom 1.8,

$$(\alpha M + (1-\alpha)m, \; \alpha M + (1-\alpha) \sim (M, m).$$

It follows by axiom 1.4 that the criterion is that of Hurwicz.

1.14 Again consider pairs (M, m). In the game given by the matrix

$$\begin{pmatrix} M & m & m \\ m & M & m \end{pmatrix},$$

since the choice set is symmetric and convex it must contain the average of the two rows which is

$$\left(\frac{M+m}{2}, \; \frac{M+m}{2}, \; m \right).$$

Therefore $(M,m) \underset{\sim}{\leq} \left(\dfrac{M+m}{2},m\right)$. By repeated application of this rule

together with the axiom of continuity we find that $(M,m) \underset{\sim}{\leq} (m,m)$.

It follows by axiom 1.4 and continuity that $(M,m) \underset{\sim}{\geq} (M',m')$ if

and only if $m \geq m'$, which is the minimax criterion.

II. Derivation of Hurwicz solution for the given matrix.

We wish to choose the a_i so as to maximize

$\propto$ Max $(a_2 + a_3 + a_4,\ a_2 + ka_4,\ a_1 + ka_2,\ a_1 + a_3 + a_4)$

$+ (1-\propto)$ Min $(a_2 + a_3 + a_4,\ a_2 + ka_4,\ a_1 + ka_2,\ a_1 + a_3 + a_4)$

$= \propto$ Max $(a_2 + a_3 + a_4,\ a_1 + a_3 + a_4) + (1-\propto)$ Min $(a_2 + ka_4,\ a_1 + ka_3)$.

This maximum can only be achieved if $a_2 + ka_4 = a_1 + ka_2$ providing

that $\propto < 1$. Assume for example that $a_2 \geq a_1$. This implies

that $a_4 = 0$ for otherwise by decreasing the value of a_4 and

increasing a_2 and a_3 we could increase the value of the expres-

sion. The equations

$$a_2 = a_1 + ka_3$$

$$a_1 + a_2 + a_3 = 0 \qquad \text{yield}$$

$$a_2 = \frac{a_1 + k - ka_1}{1+k} \qquad a_3 = \frac{1 - 2a_1}{1+k} \ .$$

This together with $0 \leq a_1 \leq a_2$ determines a line segment one end

of which is $(\tfrac{1}{2},\tfrac{1}{2},0,0)$ and the other end of which is $(0,\dfrac{k}{1+k},\dfrac{k}{1+k},0)$.

The expression is a linear function on this segment and therefore

takes its maximum either at one end or on the entire segment, as

is tabulated in section II.

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